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TOPOLOGICAL ENTROPY OF HOMOCLINIC CLOSURES

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ABSTRACT. In this paper we study the topological entropy of certain invariant sets of diffeomorphisms, namely the closure of the set of transverse homoclinic points associated with a hyperbolic periodic point, in terms of the growth rate of homoclinic orbits. First we study homoclinic closures which are hyperbolic in n-dimensional compact manifolds. Using the pseudo-orbit shadowing property of basic sets we prove a formula similar to Bowen's one on the growth of periodic points. For the nonuniformly hyperbolic case we restrict our attention to compact surfaces.

INTRODUCTION

In this paper we study the topological entropty of certain invariant sets of diffeomorphisms, namely the closure of the set of transverse homoclinic points associated with a hyperbolic periodic point. The existence of hyperbolic periodic points with transverse homoclinic points is a sufficient and necessary condition for positive entropy of C^2 diffeomorphisms of surfaces, as follows, from combining the Birkhoff-Smale Theorem [M] with Katok's work [K]. Here we introduce a way of counting homoclinic orbits, so that their growth rate yields the entropy of the closure of all transverse homoclinic points associated with a single periodic orbit of certain diffeomorphisms.

First we study homoclinic closures which are hyperbolic in n-dimensional compact manifolds; see [B] for definitions. Using the pseudo-orbit shadowing property of basic sets we prove a similar formula to Bowen's one on the growth of periodic points. For the nonuniformly hyperbolic case, we restrict our attention to compact surfaces, so we can use some of the theory developed by Katok [K] and Pesin [P].

Now we shall describe briefly how we can count homoclinic orbits. A homoclinic orbit resembles a periodic orbit of infinite period that, after a certain amount of time (both in the future and in the past), never leaves a small neighborhood of the periodic orbit. So we shall say that a homoclinic orbit is of ε-order n if n is the time during which the point wanders outside a neighborhood of size ε. In other words, if x is a homoclinic point of ε-order n, up to an ε-error, it is a periodic amount of period n. Let $f\colon M \to M$ be a diffeomorphism of a compact manifold, and suppose that $p \in M$ is a hyperbolic

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fixed point. Consider the local stable (unstable) manifold $W^s_\varepsilon(p)$ $(W^u_\varepsilon(p))$, i.e.

$$W^s_\varepsilon(p) = \{x \in M : d(f^n(x), p) \le \varepsilon \text{ for } n \ge 0\}$$
$$(W^u_\varepsilon(p) = \{x \in M : d(f^n(x), p) \le \varepsilon \text{ for } n \le 0\}).$$

Our way of counting homoclinic orbits of ε-order n can be interpreted as the number of "good" intersections of the n-iterate of $W^u_\varepsilon(p)$ with $W^s_\varepsilon(p)$. Here a point of "good" intersection means that the connected component of the unstable manifold $W^u(p)$ intersected with the ε-neighborhood containing such point is a submanifold C^1 near to $W^u_\varepsilon(p)$. This implies, in particular for diffeomorphisms of surfaces, that the 1-dimensional volume of this connected component is at least ε. This observation, together with our formula for entropy, yields an upper bound for the entropy in terms of the volume growth rate of the unstable manifolds of hyperbolic periodic points, as Newhouse [N2] has previously pointed out.

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1. HOMOCLINIC CLOSURES

Let $f: M \to M$ be a C^1 diffeomorphism of a compact manifold M. If $p \in M$ is a hyperbolic periodic point for f, the *stable* (*unstable*) *manifold of* p is defined as

$$W^s(p) = \{y \in M : d(f^n(y), f^n(p)) \to 0 \text{ as } n \to \infty\}$$
$$(W^u(p) = \{y \in M : d(f^n(y), f^n(p)) \to 0 \text{ as } n \to -\infty\}).$$

A point $x \in W^s(p) \cap W^u(p) \backslash \{p\}$ is called a *homoclinic point for* p; if the intersection of $W^s(p)$ and $W^u(p)$ is transverse at x, i.e. $T_x M = T_x W^s(p) \oplus T_x W^u(p)$, then x is called a *transverse homoclinic point.* Define

$$H(p) = \{x \in M : x \text{ is a transverse homoclinic point for } p\},$$

and if $\mathbf{O}(p)$ denotes the orbit of p and n its period set $H(\mathbf{O}(p)) = H(p) \cup \cdots \cup H(f^{n-1}(p))$. We shall call the closure of $H(p)$, $\mathrm{CL}\,H(p)$, the *homoclinic closure of* p, and $\mathrm{CL}\,H(\mathbf{O}(p))$ the *homoclinic closure of* $\mathbf{O}(p)$.

Clearly, $\mathrm{CL}\,H(\mathbf{O}(p))$ is compact and f-invariant. Furthermore, the λ-lemma [N1] shows that $f|\mathrm{CL}\,H(\mathbf{O}(p))$ is transitive (there is a dense orbit in $\mathrm{CL}\,H(\mathbf{O}(p))$), and $f^n|\mathrm{CL}\,H(p)$ is topological mixing (for any open sets $U, V \subset \mathrm{CL}\,H(p)$ there exists N so that $f^{nk}U \cap V \ne \varnothing$ for any $k \ge N$). The Birkhoff-Smale Theorem on homoclinic points implies that hyperbolic periodic points are dense in $\mathrm{CL}\,H(\mathbf{O}(p))$.

We shall say that two hyperbolic periodic points $p, p' \in M$ are *homoclinically related* if $W^s(p)$ intersects $W^u(p')$ transversally and $W^u(p)$ intersects $W^s(p')$ transversally. This is clearly an equivalence relation; see [N1].

2. Counting homoclinic orbits

Now fix $\varepsilon > 0$ small. If $H(p) \neq \emptyset$ let us define the *ε-order* $\Theta(x, p, \varepsilon)$ of $x \in H(p)$ with respect to p as follows: Let

$$\Theta^s(x, p, \varepsilon) = \min\{n: f^n(x) \in W^s_\varepsilon(p)\},$$
$$\Theta^u(x, p, \varepsilon) = \min\{n: f^{-n}(x) \in W^u_\varepsilon(p)\}, \quad \text{and}$$
$$\Theta(x, p, \varepsilon) = \Theta^s(x, p, \varepsilon) + \Theta^u(x, p, \varepsilon).$$

If $\Theta(x, p, \varepsilon) = n$, then x is said to be a homoclinic point of ε-order n. Now let

$$H(p, n, \varepsilon) = \{x \in H(p): \Theta^s(x, p, \varepsilon) = n \text{ and } \Theta^u(x, p, \varepsilon) = 0\},$$

and write $h(p, n, \varepsilon)$ for $\#H(p, n, \varepsilon)$. One would like $h(p, n, \varepsilon)$ to be finite, but this is not true in many cases; for instance if we start with the figure 8 flow φ_t on R^2 and consider $f = \varphi_1$ its time-1 map, or any diffeomorphism with the stable and unstable manifold of a fixed point intersecting tangentially. One can make a small local perturbation of f in any C^r topology, $1 \leq r < \infty$, so that the unstable manifold of p locally looks like $x \to x^{r+1}\sin(1/x)$ over an interval of the stable manifold of p. The perturbed diffeomorphism g can be made to still fix p, and for any small ε clearly $h(p, n, \varepsilon) = \infty$ for g.

The above example does not satisfy the definition of hyperbolicity. If the homoclinic closure is hyperbolic, then the density of periodic points implies that the diffeomorphism f restricted to $\mathrm{CL}\,H(\mathbf{O}(p))$ is expansive, i.e. there exists $\alpha > 0$ such that if $d(f^n(x), f^n(y)) < \alpha$ for all n, then $x = y$. So if $0 < \varepsilon < \alpha$ and $x \neq y \in H(p, n, \varepsilon)$, then expansiveness implies the existence of $0 < k < n$ such that $d(f^n(x), f^n(y)) > \varepsilon$, which, together with compactness, implies that $H(p, n, \varepsilon)$ is finite. Thus, in the case of hyperbolic homoclinic closures, it makes sense to consider the growth rate of $h(p, n, \varepsilon)$.

Now we shall discuss how to count homoclinic orbits for diffeomorphisms of surfaces whose homoclinic closures may not satisfy the hyperbolicity conditions. Let $f: M \to M$ be a diffeomorphism of a surface M. Assume that p is a hyperbolic fixed point for f with eigenvalues λ_1, λ_2 satisfying $|\lambda_1| < 1 < |\lambda_2|$. Set

$$\chi(p) = (1987/2000)\max\{-\log|\lambda_1|, \log|\lambda_2|\};$$

then for any $x \in H(p)$ and $0 < \chi \leq \chi(p)$, there exists $C > 1$ such that for $n > 0$, $m \in Z$ we have:
For $v \in D_x f^m | T_x W^s(p)$,

$$\|D_{f^m(x)} f^n v\| \leq C\exp(-n\chi)\exp(\chi 10^{-3}(|m| + n))\|v\|, \tag{1s}$$

$$\|D_{f^m(x)} f^{-n} v\| \geq C^{-1}\exp n\chi \exp(-\chi 10^{-3}(|m| + n))\|v\|; \tag{2s}$$

for $v \in D_x f^m | T_x W^u(p)$,

(1u) $$\|D_{f^m(x)} f^n v\| \geq C^{-1} \exp n\chi \exp(-\chi 10^{-3}(|m| + n))\|v\|,$$

(2u) $$\|D_{f^m(x)} f^{-n} v\| \leq C \exp(-n\chi) \exp(\chi 10^{-3}(|m| + n))\|v\|;$$

and if $\gamma(x)$ denotes the angle between the subspaces $T_x W^s(p)$ and $T_x W^u(p)$, then

$$\gamma(f^m(x)) \leq C^{-1} \exp(-\chi) 10^{-3}|m|.$$

These conditions are clear consequences of the λ-lemma, since we can take a small neighborhood R of p such that, for any $x \in H(p)$, there exists $n > 0$ so that, for any $k > n$, we have:
For $v \in D_{f^k(x)} T_x W^u(p)$,

$$\|D_{f^k(x)} f v\| \geq \exp \chi(p) \|v\|,$$

$$\|D_{f^{-k}(x)} f^{-1} v\| \leq \exp(-\chi(p)\|v\|);$$

similarly, for $v \in D_{f^{-k}(x)} T_x W^s(p)$,

$$\|D_{f^k(x)} f v\| \leq \exp \chi(p) \|v\|,$$

$$\|D_{f^{-k}(x)} f^{-1} v\| \geq \exp(-\chi(p)\|v\|);$$

and the angle between the subspaces satisfies $\gamma(f^k(x)) > \gamma(p) 1987/2000$. Since x only spends a finite amount of time out or R, we can find $C(x)$ to satisfy the above conditions. This shows that $\mathrm{CL}\,\mathbf{O}(x) = \mathbf{O}(x) \cup \{p\}$ is a hyperbolic set, although it does not have local product structure. Conditions (1s), (2s), (1u), (2u) are, in fact, more general than what we have just shown; they correspond to the definition of nonuniform hyperbolicity. For $0 < \chi \leq \chi(p)$, $C > 1$, write $\Lambda_{\chi, C}$ for the set of points of M satisfying conditions (1s), (2s), (1u), (2u). Set

$$H(p, n, \varepsilon, \chi, C) = \{x \in H(p, n, \varepsilon) \cap \Lambda_{\chi, C} : f^n(x) \in \Lambda_{\chi, C}\}$$

and write $h(p, n, \varepsilon, \chi, C)$ for $\# H(p, n, \varepsilon, \chi, C)$.

3. The hyperbolic case

In this section we shall assume that $\mathrm{CL}\, H(\mathbf{O}(p))$ is a hyperbolic set; since the hyperbolic periodic points are dense in $\mathrm{CL}\, H(\mathbf{O}(p))$ it has local product structure and it can be regarded as a basic set of an Axiom A diffeomorphism.

We recall that the topological entropy for a homeomorphism $T: X \to X$ of a compact metric space X is defined as follows: For $\varepsilon > 0$, $n > 0$ a subset $E \subset X$ is said to be *(n, ε)-separated* if for any $x, y \in E$ there exists $0 \leq k \leq n$ for which $d(f^k(x), f^k(y)) > \varepsilon$. Let $s(n, \varepsilon)$ be the maximal cardinality of (n, ε)-separated subsets of X, then defined the *topological entropy of T* as

$$h(T) = h(T, X) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} (1/n) \log s(n, \varepsilon).$$

If $Y \subset X$ is a T-invariant subset set $h(T, Y) = h(T|Y, Y)$.

Theorem 3.1. *Let $f: M \to M$ be a C^1 diffeomorphism of a compact manifold M. If $p \in M$ is a hyperbolic periodic point for f such that $H(p) \neq \emptyset$ and $\mathrm{CL}\,H(\mathbf{O}(p))$ is a hyperbolic set for f, then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$*

$$h(f, \mathrm{CL}\,H(\mathbf{O}(p))) = \limsup_{n\to\infty}(1/n)\log h(p, n, \varepsilon).$$

Furthermore if p is a fixed point, then

$$h(f, \mathrm{CL}\,H(p)) = \lim_{n\to\infty}(1/n)\log h(p, n, \varepsilon) = \sup_n(1/n)\log h(p, n, \varepsilon).$$

Proof. Assume that p is fixed. The stable manifold theorem implies that $f|\mathrm{CL}\,H(p)$ is expansive, that there exists $\delta > 0$ such that if $x, y \in \mathrm{CL}\,H(p)$ and $d(f^n(x), f^n(y)) \leq \delta$ for all $n \in Z$, then $x = y$.

Now choose $0 < \varepsilon \leq \delta/4 = \varepsilon_0$ and $x, y \in H(p, n, \varepsilon)$. Observe that expansiveness implies that there exists $0 < k < n$ so that $d(f^n(x), f^n)(y)) > \varepsilon$. Therefore, $H(p, n, \varepsilon)$ is an (n, ε)-separated set for $f|\mathrm{CL}\,H(p)$, and thus

$$\limsup_{n\to\infty}(1/n)\log h(p, n, \varepsilon) \leq h(f, \mathrm{CL}\,H(p)).$$

To complete the proof of the theorem we shall use Bowen's method of pseudo-orbit shadowing [B]. A sequence $\underline{x} = \{x_i\}_{i=a}^{b}$ (with $a = -\infty$ or $b = \infty$ permitted) of points in M is an *α-pseudo-orbit*, for some $\alpha > 0$, if $d(f(x_i), x_{i+1}) < \alpha$ for all $i \in [a, b)$. A point $x \in M$ is said to be *β-shadow* $\underline{x}$, for $\beta > 0$, if

$$d(f^i(x), x_i) \leq \beta \quad \text{for all } i \in [a, b).$$

Lemma 3.3 (Shadowing Lemma [B, N1]). *Let $\mathrm{CL}\,H(p)$ be a hyperbolic set for f, then for every $\beta > 0$ there exists $\alpha > 0$ so that every α-pseudo orbit x in $\mathrm{CL}\,H(p)$ is β-shadowed by a point $x \in \mathrm{CL}\,H(p)$.* □

To complete the proof of the theorem we shall shadow some sets of separated points by homoclinic orbits. First observe that if $x \in H(p, n, \varepsilon)$, then

$$x \in \mathrm{CL}(W^u_\varepsilon(p)\backslash f^{-1}W^u_\varepsilon(p)) \quad \text{and} \quad f^n(x) \in \mathrm{CL}(W^s_\varepsilon(p)\backslash fW^s_\varepsilon(p)).$$

Let $0 < \beta < \varepsilon/8$ and α be as in the Shadowing Lemma. By compactness of $\mathrm{CL}\,H(p)$ we can take a finite cover U of $\mathrm{CL}\,H(p)$ by α-balls. Since $f|\mathrm{CL}\,H(p)$ is topological mixing, there exists $N = N(U) > 0$ such that, for any two balls $B_i, B_j \in U$,

$$f^{-k}B_i \cap B_j \cap \mathrm{CL}\,H(p) \neq \emptyset \quad \forall k \geq N.$$

Choose $w_1 \in (W^s_\varepsilon(p)\backslash fW^s_\varepsilon(p)) \cap H(p)$ and $w_2 \in (W^u_\varepsilon(p)\backslash f^{-1}W^u_\varepsilon(p)) \cap H(p)$, and assume that $B(w_1, \alpha)$ and $B(w_2, \alpha)$ belong to the cover U. Now let E be a (n, ε)-separated set in $\mathrm{CL}\,H(p)$ of maximal cardinality. For $x \in E$

take $u(x) \in f^{-N}B(x, \alpha) \cap B(w_1, \alpha) \cap \mathrm{CL}\, H(p)$ and $v(x) \in f^{N}B(f^n(x), \alpha) \cap B(w_2, \alpha) \cap \mathrm{CL}\, H(p)$ (we shall assume that f preserves orientation), and consider the following sequence:

$$\ldots, f^{-1}(w_2), u(x), f(u(x)), \ldots, f^{N-1}(u(x)), x, f(x), \ldots, \\ f^{n-1}(x), f^{-N}(v(x)), \ldots, f^{-1}(v(x)), w_1, f(w_1), \ldots;$$

by construction the above sequence is an α-pseudo-orbit contained in $\mathrm{CL}\, H(p)$, and by the Shadowing Lemma there exists $z(x) \in \mathrm{CL}\, H(p)$ that β shadows it.

Now we show that $z(x) \in H(p)$. If $d(x, z(x)) < \beta$ then

$$d(f^k(w_1), f^{k+n+N}(z(x))) < \beta \quad \text{for all } k \geq 0,$$

which implies that $f^k(z(x)) \to p$ as $k \to \infty$. Similarly $f^{-k}(z(x)) \to p$ as $k \to \infty$, thus $z(x) \in H(p)$; furthermore if α is chosen small enough the hyperbolic structure around p implies that $\Theta(z(x), p, \varepsilon) = 2N + n$.

If $x, y \in E$, then $z(x) \neq z(y)$ because

$$0 < \varepsilon - 2\beta < \varepsilon - d(f^k(x), f^k(z(x))) - d(f^k(y), f^k(z(y))) \\ \leq d(f^k(z(x)), f^k(z(y)))$$

for some $k \in [0, n)$, since E is (n, ε)-separated. Therefore $s(n, \varepsilon) \leq h(p, 2N + n, \varepsilon)$ for all $n \geq 0$ and N is independent of n and ε. Hence by expansiveness

$$h(f, \mathrm{CL}\, H(p)) = \limsup_{n\to\infty} (1/n) \log h(p, n, \varepsilon).$$

To prove the existence of such a limit observe that by considering the intersection of local stable manifolds of points in $H(p, n, \varepsilon)$ with the local unstable manifolds of $f^m H(p, m, \varepsilon)$ it follows that $h(p, n, \varepsilon)\ h(p, m, \varepsilon) \leq h(p, n+m, \varepsilon)$, then a standard argument shows that if p is fixed then

$$h(f, \mathrm{CL}\, H(p)) = \lim_{n\to\infty} (1/n) \log h(p, n, \varepsilon) = \sup_n (1/n) \log h(p, n, \varepsilon).$$

If p is not fixed take a power f^k corresponding to its period and notice that the ε-order $\Theta_k(x, p, \varepsilon)$ calculated with respect to f^k satisfies

$$\Theta_k(x, p, \varepsilon) = (1/k)\Theta(x, p, \varepsilon). \quad \square$$

4. Homoclinic closures for diffeomorphisms of surfaces

In this section we shall study homoclinic closures for C^2 diffeomorphisms of surfaces that are not necessarily hyperbolic sets. The main tool here will be the following statement due to Katok [K, KM].

Theorem 4.1 (Katok). *Let $f: M \to M$ be a C^2 diffeomorphism of a compact surface M; then*

$$h(f) = \sup\{h(f, \Lambda): \Lambda \textit{ is a hyperbolic horseshoe}\}. \quad \square$$

In order to avoid some technical difficulties we shall restrict our attention to isolated homoclinic closures, that is, $\mathrm{CL}\,H(p)$ is *isolated* if there exists an open set U such that $\mathrm{CL}\,H(p) \subset U$ and $U \cap \Omega(f) = \mathrm{CL}\,H(p)$, where $\Omega(f)$ denotes the nonwandering set of f. Some examples of nonhyperbolic isolated homoclinic closures will appear in [KM].

Theorem 4.2. *Let $f\colon M \to M$ be a C^2 diffeomorphism of a compact surface M, and let $p \in M$ be a hyperbolic periodic point for f with $H(p) \neq \varnothing$. Then for $0 < \chi$ small*

$$h(f\,,\,\mathrm{CL}\,H(p)) = \sup_{C>1} \lim_{\varepsilon\to 0} \limsup_{n\to\infty} (1/n) \log h(p\,,n\,,\varepsilon\,,\chi\,,C).$$

Part of the proof of the above theorem is based on the following

Proposition 4.3. *Let $f\colon M \to M$ be a C^2 diffeomorphism of a compact surface M, and let $p \in M$ be a hyperbolic periodic point for f with $H(p) \neq \varnothing$. Then for $0 < \chi < \chi(p)$ and $C = C(p) > 1$ there exists $\varepsilon_0 = \varepsilon_0(\chi\,,C) > 0$ such that for $0 < \varepsilon < \varepsilon_0$*

$$\limsup_{n\to\infty} (1/n) \log h(p\,,n\,,\varepsilon\,,\chi\,,C) \leq h(f\,,\,\mathrm{CL}\,H(p)).$$

Before giving the proof of Theorem 4.2 and Proposition 4.3 we shall recall some facts and definitions from smooth ergodic theory; see [K, KM and P]. For $f\colon M \to M$ a C^2 diffeomorphism of a compact surface M and $0 < \chi$, $1 < C$, consider the set $\Lambda_{\chi\,,C}$. Standard arguments show that $\Lambda_{\chi\,,C}$ is closed and the splitting of the tangent space varies continuously on it. Set $\Lambda_\chi = \bigcup_{C>1} \Lambda_{\chi\,,C}$.

Proposition 4.5 (Existence of regular neighborhoods). *For $x \in \Lambda_\chi$, there exists $\delta(x) > 0$ and a C^1 embedding*

$$\psi_x\colon [-\delta(x)\,,\delta(x)] \times [-\delta(x)\,,\delta(x)] \to M$$

satisfying:

(i) $\delta(x) \geq \delta(f^m(x)) \max\{(1.5)^{-m}\,,\ \exp(-2\chi 10^{-3}|m|)\}$.

(ii) *If $C > 1$, then $\delta_{\chi\,,C} = \min\{\delta(x)\colon x \in \Lambda_{\chi\,,C}\} > 0$.*

(iii) *The map $f_x = \psi_{f(x)}^{-1} \circ f \circ \psi_x\colon [-\delta(x)\,,\delta(x)]^2 \to R^2$ is hyperbolic.* □

For $x \in \Lambda_\chi$ and $0 < t \leq 1$ we shall call $R(x\,,t) = \psi_x([-t\delta(x)\,,t\delta(x)]^2)$ the *t-regular neighborhood of* x. Also, for $0 < \alpha < \frac{1}{2}$, we shall call any manifold $W \cap R(x\,,t)$ an *admissible $(u\,,\alpha\,,t)$-manifold near* x if $W = \psi_x\,\mathrm{Graph}_\varphi$, where $\varphi\colon [-t\delta(x)\,,t\delta(x)] \to [-t\delta(x)\,,t\delta(x)]$ is a C^1 map with $\|D\varphi\| \leq \alpha < \frac{1}{2}$. Similarly define *admissible $(s\,,\alpha\,,t)$-manifold near* x. A sort of λ-lemma argument show that f maps admissible $(u\,,\alpha\,,t)$-manifold near x onto admissible $(u\,,\alpha\,,t)$-manifold near $f(x)$. An *admissible $(u\,,t)$-rectangle U near* x is defined as

$$\psi_x(\{(z\,,y) \in [-t\delta(x)\,,t\delta(x)]^2\colon \varphi_1(y) \leq z \leq \varphi_2(y)\}),$$

where $\varphi_i\colon [-t\delta(x), t\delta(x)] \to [-t\delta(x), t\delta(x)]$, $i = 1, 2$, are C^1 maps with $\|D\varphi_i\| \leq \alpha < \frac{1}{2}$. Similarly, define *admissible* (s, t)*-rectangles* as the set contained between two admissible (s, α, t)-manifolds.

Proof of Proposition 4.3. Suppose that p is a fixed point for f and choose $C > 1$ and $0 < \chi < \chi(p)$. Observe that $p \in \Lambda_{\chi, C}$. Consider the regular neighborhoods $R(p, t)$ of p, with $t > 0$ so small so that, for $x \in \Lambda_{\chi, C} \cap R(p, t)$, then $C(x, W^s(x) \cap R(p, t))$ $(C(x, W^u(x) \cap R(p, t))$, the connected component of $W^s(x) \cap R(p, t)$ $(W^u(x) \cap R(p, t))$ containing x, is an admissible $(s, \frac{1}{2}, t)$-manifold $((u, \frac{1}{2}, t)$-manifold) near p. This is possible since $\Lambda_{\chi, C}$ is "uniformly hyperbolic", although possibly not f-invariant. Thus if $x \in H(p, n, \varepsilon, \chi, C)$ with $0 < \varepsilon < t\delta_{\chi, c}/2$, then by [K]

$$C(f^n(x), f^n R(p, t) \cap R(p, t))$$

is an admissible (u, t)-rectangle in $R(p, t)$. Similarly,

$$f^{-n}(C(f^n(x), f^n(R(p, t) \cap R(p, t)))$$

results in an admissible (s, t)-rectangle in $R(p, t)$. Now let

$$D_x^s = C(x, W^s(x) \cap R(p, t))$$

and

$$D_{f^n(x)}^u = (f^n(x), W^u(x)) \cap R(p, t)).$$

We may assume that for all $x \in H(p, n, \varepsilon, \chi, C)$ we have that $f^n D_x^s \subset W_\varepsilon^s(p)$ and $f^{-n} D_{f^n(x)}^u$, then if $x \neq y \in H(p, n, \varepsilon, \chi, C)$ clearly $D_x^s \cap D_y^s = \varnothing$ and $D_{f^n(x)}^u \cap D_{f^n(y)}^u = \varnothing$, which implies that

$$C(f^n(x), f^n R(p, t) \cap R(p, t)) \cap C(f^n(x), f^n R(p, t) \cap R(p, t)) = \varnothing.$$

Thus we have $h(p, n, \varepsilon, \chi, C)$ admissible (s, t)-rectangles mapped by f^n onto the same number of admissible (u, t)-rectangles, then standard arguments [M] show the existence of an $h(p, n, \varepsilon, \chi, C)$-fold horseshoe of period n, whose topological intropy is $1/n \log h(p, n, \varepsilon, \chi, C)$. □

Proof of Theorem 4.2. Without loss of generality let us suppose that p is a fixed point. By Proposition 4.3 it is sufficient to prove that for any $\beta > 0$ there exist a small $\varepsilon > 0$, $\chi > 0$ and $C > 1$ so that

$$h(f, \mathrm{CL}\, H(p)) \leq \limsup_{n\to\infty} (1/n) \log h(p, n, \varepsilon, \chi, C) + \beta.$$

The idea of the proof is to use Theorem 4.1 to find a horseshoe $\Lambda \subset \mathrm{CL}\, H(p)$ with entropy $h(f, \Lambda) \geq h(f, \mathrm{CL}\, H(p)) - \beta$. Then any periodic point $p' \in \Lambda$ is homoclinically related to p; thus, for $\rho = \rho(\Lambda) > 0$ we can choose $m > 0$ sufficiently large and small disks $D^u \subset W_\varepsilon^u(p) \backslash f^{-1}(W_\varepsilon^u(p))$ and $D^s \subset W_\varepsilon^s(p) \backslash f(W_\varepsilon^s(p))$ so that $f^m D^u$ and $f^{-m} D^s$ are sufficiently C^1 close to $W_\rho^u(p')$ and $W_\rho^s(p')$, respectively. Then, using the hyperbolicity of Λ, for each homoclinic point $y \in H(p', n, \rho) \cap \Lambda$ we construct a transverse homoclinic point $x = x(y)$ of p with $\Theta(p, n+2m, \varepsilon)$. The hyperbolicity along the orbit of x is determined basically by that for y and by $D_p f$. The sets $H(p', n, \rho) \cap \Lambda$ will play the role of the (n, ε)-separated sets of the proof of Theorem 3.1 and the points $x = x(y)$, for $y \in H(p', n, \rho) \cap \Lambda$, will shadow the orbit of y up to n iterates.

By Theorem 4.1 for $\beta > 0$ fixed, choose a horseshoe $\Lambda \subset \mathrm{CL}\, H(p)$ such that

$$h(\mathrm{CL}\, H(p), f) \leq h(\Lambda, f) + \beta.$$

We may assume that Λ contains a fixed point p' and that there exists $\chi_1 > h(f, \Lambda)$ such that for $y \in \Lambda$, $n \geq 0$

$$\|D_y f^n v\| \leq \exp(-\chi_1 \|v\|) \quad \text{for } v \in E^s_y$$

and

$$\|D_y f^n v\| \geq \exp(-\chi_1 \|v\|) \quad \text{for } v \in E^u_y.$$

This is possible by taking a large power of f.

Let $\varepsilon > 0$ be so small that if $x \in H(p, n, \varepsilon)$, for $n \geq 0$, there exists $C > 1$ such that $x \in H(p, n, \varepsilon, \chi(p), C)$. Choose $\rho = \rho(\Lambda) > 0$ for which $f|\Lambda$ is expansive. Since p is homoclinically related to p', the λ-lemma [N1] enables us to choose $m > 0$, $D^u \subset W^u_\varepsilon(p) \backslash f^{-1} W^u_\varepsilon(p)$ and $D^s \subset W^u_\varepsilon(p) \backslash W^s_\varepsilon(p)$ such that $f^m D^u$ and $f^{-m} D^s$ are as close as we wish to $W^u_\rho(p')$ and $W^s_\rho(p')$, respectively, in the C^1 sense.

Let us recall some facts that would tell us how close $f^m D^u$ and $f^{-m} D^u$ should be to $W^u_\rho(p')$ and $W^s_\rho(p')$, respectively, to allow us to construct the new homoclinic points.

Since f is C^2 for a fixed $\delta > 0$ we can choose $a > 0$ and $\alpha > 0$ such that if $y \in \Lambda$, $z \in M$ and $d(z, y) < a$, then if W is a $(u, \alpha, 1)$-admissible manifold near y and $z \in W$

$$|\, \|D_y f | E^u_y\| - \|D_z f | T_z W\| \,| < \delta.$$

And if W is an $(s, \alpha, 1)$-admissible manifold near y and $z \in W$

$$|\, \|D_y f | E^s_y\| - \|D_z f | T_z W\| \,| < \delta.$$

The uniform hyperbolicity of Λ implies that for any $0 < t \leq 1$ the size of the regular neighborhood $R(y, t)$, $y \in \Lambda$, is constant on Λ. Suppose that $\rho/2 > a > 0$ is so small that $W^u_a(y) \subset R(y, 1)$ for any $y \in \Lambda$, and that the choice of $m > 0$ and $D^u \subset W^u_\varepsilon(p) \backslash f^{-1} W^u_\varepsilon(p)$ is so that $f^m D^u$ is C^1 close enough to $W^u_\rho(p')$ that for any $y \in W^u_\rho(p') \cap \Lambda$, $W^s_a(y) \cap f^m D^u \neq \emptyset$ and $f^m D^u \cap R(y, t)$ is an admissible (u, α, t)-manifold near y, for some fixed $0 < t \leq 1$ and $\alpha > 0$ small. Choose $D^s \subset W^s_\varepsilon(p) \backslash f W^s_\varepsilon(p)$ satisfying similar conditions.

By taking a power of f we may assume that p' is fixed; therefore,

$$\lim_{n\to\infty}(1/n)\log\#(H(p',n,\rho)\cap\Lambda)=h(f,\Lambda).$$

Thus if $y\in H(p',n,\rho)\cap\Lambda$, consider $z=z(y)\in W_a^s(y)\cap f^mD^u$ and $V_0(y)=f^mD^u\cap R(y,t)$. Define

$$\begin{aligned} V_1(y)&=f(V_0(y))\cap R(f(y),t),\\ V_2(y)&=f(V_1(y))\cap R(f^2(y),t),\\ &\vdots\\ V_n(y)&=f(V_{n-1}(y))\cap R(f^n(y),t). \end{aligned}$$

For each $0\le k\le n$, $V_k(y)$ is an admissible (u,α,t)-manifold near $f^k(y)$. Therefore $V_n(y)$ intersects $f^{-m}D^s$ transversely at a point $w=w(y)\in H(p)$, so let $x=x(y)=f^{-n}w$. For $t>0$ sufficiently small $d(f^k(y),f^k(x(y)))<a$ for all $0\le k\le n$.

By construction $\Theta(x,p,\varepsilon)=n+2m$, and m is clearly independent of n. So it remains to find $\chi>0$ and $C>1$ such that for any $x=x(y)\in H(p)$ constructed as above for $y\in H(p',n,\rho)\cap\Lambda$, $f^{-m}(x)\in\Lambda_{\chi,C}$ and $f^{m+n}(x)\in\Lambda_{\chi,C}$ and to check that

$$\#(H(p',n,\rho)\cap\Lambda)\le h(p,n+2m,\varepsilon,\chi,C),$$

from which the theorem follows.

For $x\in H(p)$ let $E_x^u=T_xW^u(p)$ and $E_x^s=T_xW^s(p)$, by transversality $T_xM=E_x^s\oplus E_x^u$. Recall that $\gamma(x)$ denotes the angle between the subspaces E_x^s and E_x^u. By similar argument to the proof of the λ-lemma [N1], if $\varepsilon>0$ is sufficiently small, if $x\in H(p)\cap W_\varepsilon^s(p)$ and $\gamma(x)\ge s$ for some $s>0$, then $\gamma(f^k(x))\ge s$ for all $k\ge 0$. Similarly if $x\in H(p)\cap W_\varepsilon^u(p)$ and $\gamma(x)\ge s$, then $\gamma(f^{-k}(x))\ge s$ for all $k\ge 0$.

Now for $x=x(y)\in H(p)$, $y\in H(p',n,\rho)\cap\Lambda$, the angle $\gamma(f^k(x))$ is bounded away from zero, say greater than or equal to some $r>0$, independent of n, for all $0\le k\le n$. This is so because $f^k(x)$ is the intersection of two admissible manifolds near $f^k(x)$. By linear algebra we have that for any $0<k\le m$

$$\operatorname{sen}\gamma(f^{-k}(x))\ge(\operatorname{sen}r)\left(\inf_{w\in M}|\det D_wf|\right)^m\left(\sup_{w\in M}\|D_wf\|\right)^{-2m}$$

and

$$\operatorname{sen}\gamma(f^{n+k}(x))\ge(\operatorname{sen}r)\left(\inf_{w\in M}|\det D_wf|\right)^m\left(\sup_{w\in M}\|D_wf\|\right)^{-2m}.$$

Since $f^{-m}(x)\in W_\varepsilon^u(p)$ and $f^{m+n}(x)\in W_\varepsilon^s(p)$, there exists $C_1>1$ such that $\gamma(f^i(x))\ge C_1^{-1}$ for all $i\in Z$.

Now we shall look at how the derivative acts on E^s_x, for $x = x(y)$ and $y \in H(p', n, \rho) \cap \Lambda$. Take $v \in E^s_x$, since $d(f^k(x), f^k(y)) < a$ for $0 \le k \le n$, then it follows that if $\delta \le \exp(-\chi_1)\exp(10^{-3} - 1)$

$$\begin{aligned}\|D_x f^k v\| &\le \left(\prod_{i=0}^{k-1} \|D_{f^i(x)} f | E^s_{f_i(x)}\|\right) \|v\| \\ &\le \left(\prod_{i=0}^{k-1} (\|D_{f^i(y)} f | E^s_{f^i(y)}\| + \delta)\right) \|v\| \\ &\le \exp(-(\chi_1 - 10^{-3}))k\|v\|.\end{aligned}$$

If $\varepsilon > 0$ is small and $k \ge 0$, for $v \in E^s_{f^{m+n}(x)}$

$$\|D_{f^{m+n}(x)} f^k v\| \le \exp(-(\chi(p) - 10^{-3}))k\|v\|$$

and for $v \in E^s_{f^{-m}(x)}$

$$\|D_{f^{-m}(x)} f^{-k} v\| \ge \exp(\chi(p) - 10^{-3})k\|v\|.$$

And for $0 < k \le m$, if $D = \sup\{\|D_W f\| : w \in M\}$, then

$$\|D_{f^n(x)} f^k v\| \le D^k \|v\| \quad \text{and} \quad \|D_x f^{-k} v\| \ge D^{-k}\|v\|.$$

Similar inequalities hold for $v \in E^u_x$. Now let us set $\chi = \min\{\chi_1, \chi(p)\}$ and $C = \max\{C_1, (D\exp(\chi - 10^{-3}))^{2m}\}$. Using the above inequalities it is straightforward to check that if $y \in H(p', n, \rho) \cap \Lambda$ for any $n > 0$, then $x = x(y) \in \Lambda_{\chi, C}$ and so do $f^{-m}(x)$ and $f^{m+n}(x)$. Since $\Theta^u(f^{-m}(x), p, \varepsilon) = 0$ and $\Theta^s(f^{-m}(x), p, \varepsilon) = 2m+n$, $f^{-m}(x) \in H(p, n+2m, \varepsilon, \chi, C)$. It remains to prove that

$$\#(H(p', n, \rho) \cap \Lambda) \le h(p, n + 2m, \varepsilon, \chi, C).$$

If $y_1, y_2 \in 2H(p', n, \rho) \cap \Lambda$, then $d(f^k(y_1), f^k(y_2)) > \rho$ for some $0 \le k \le n$ (by expansiveness of $f|\Lambda$), so

$$\begin{aligned}\rho \le d(f^k(y_1), f^k(y_2)) &\le d(f^k(y_1), f^k(x(y_1)) + d(f^k(x(y_1)), f^k(x(y_2))) \\ &\quad + d(f^k(x(y_2)), f^k(y_2)) \\ &\le 2a + d(f^k(x(y_1)), f^k(x(y_2))).\end{aligned}$$

Hence since $2a < \rho$, it follows that $x(y_1) \ne x(y_2)$.

Now by taking limits where $n \to \infty$, we have

$$h(f, \Lambda) = \lim_{n\to\infty} (1/n)\log\#(H(p', n, \rho) \cap \Lambda) \le \limsup_{n\to\infty}(1/n)\log h(p, n, \varepsilon, \chi, C)$$

from which the theorem follows. □

5. Entropy and volume

Let $f: M \to M$ be a C^1 diffeomorphism of a compact surface M. For a 1-dimensional submanifold $W \subset M$ define its *volume growth* as

$$\log \mathrm{Vol}(f|W) = \limsup_{n\to\infty} (1/n) \log \mathrm{Vol}(T_{f^n}|W),$$

where $T_{f^n}|W \subset W \times M$ is the graph of the nth iterate of W under f and Vol denotes the 1-dimensional Riemannian volume or length. See [G, **N2** and **Y**] for more details.

Theorem 5.1. *Let $f: M \to M$ be a C^2 diffeomorphism of a compact surface M and suppose that $p \in M$ is a hyperbolic fixed point for f such that $\mathrm{CL}\, H(p) \neq \emptyset$, then*

$$h(f, \mathrm{CL}\, H(p)) \leq \inf\{\log \mathrm{Vol}(f|W_\varepsilon^u(p)) : \varepsilon > 0\}.$$

Corollary 5.2. *For f as above,*

$h(f) \leq \sup\{\inf\{\log \mathrm{Vol}(f|W_\varepsilon^u(p)) : \varepsilon > 0\} : p$ *is a hyperbolic periodic point*$\}$. □

Proof of Theorem 5.1. The number $h(p, n, \varepsilon, \chi, C)$ means that we have $h(p, n, \varepsilon, \chi, C)$ admissible u-manifolds near p with length at least ε, so

$$h(p, n, \varepsilon, \chi, C) \times \varepsilon \leq \mathrm{Vol}(T_{f^n}|W_\varepsilon^u(p)),$$

from which the theorem follows. □

References

[B] R. Bowen, *On Axiom* A *diffeomorphisms*, CBMS Regional Conf. Ser. Math., no. 35, Amer. Math. Soc., Providence, R.I., 1978.

[G] M. Gromov, *Entropy, homology and semialgebraic geometry*, Astérisque **145–6** (1987), 225–240.

[K] A. B. Katok, *Lyapunov enponents, entropy and periodic points*, Inst. Hautes Études. Sci. Publ. Math. **51** (1980), 137–173.

[KM] A. B. Katok and L. Mendoza, *Smooth ergodic theory*, in preparation.

[M] J. Moser, *Stable and random motions in dynamical systems*, Ann. of Math. Stud., Princeton Univ. Press, Princeton, N.J., 1973.

[N1] S. Newhouse, *Lectures on dynamical systems*, Birkhäuser, Boston, Mass., 1980.

[N2] ——, *Entropy and volume*, Preprint.

[P] Ya. B. Pesin, *Characteristic Lyapunov exponents and smooth ergodic theory*, Russian Math. Surveys **32** (1977), 55–114.

[Y] Y. Yomdin, *Volume growth and entropy*, Preprint, Inst. Hautes Études Sci., 1986.

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